QED one-loop corrections to a macroscopic magnetic dipole

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Abstract.   We consider the field equations of a static magnetic field including one-loop QED corrections, and calculate the corrections to the field of a magnetic dipole.

1. Introduction

The one-loop corrections of quantum electrodynamics introduce nonlinearities in the equations of the electromagnetic field. These corrections manifest themselves through an index of refraction, electric permittivity and magnetic permeability tensors which are a function of field strength [1, 2]. The vacuum responds to an applied field like a non-linear paramagnetic substance [3]. Using an analytic expression for the effective Lagrangian of QED to one-loop order [4], we derive the magnetic permeability tensor as a function of the applied magnetic field and calculate the one-loop corrections to the field of a macroscopic magnetic dipole.

2. The effects of non-linearity on the field equations

The effective Lagrangian of QED at one-loop order consists of the sum of a linear and a non-linear term

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1. \tag{1}$$

Both terms of the Lagrangian can be written in terms of the Lorentz invariants,

$$I = F_{\mu\nu}F^{\mu\nu} = 2(|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2) \tag{2}$$

and

$$K = -(\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}F_{\lambda\rho}F_{\mu\nu})^2 = -(4\boldsymbol{E}\cdot\boldsymbol{B})^2 \tag{3}$$

following Heisenberg and Euler [5]. The Greek indices count over space and time components (0, 1, 2, 3).

Since we are interested in the static properties of a magnetic field, we can take $K = 0$. If we apply the Euler–Lagrange condition to extremize the action, we obtain,

$$\nabla \times \boldsymbol{H} = 0 \tag{4}$$

$$\boldsymbol{H} = -4\frac{\partial\mathcal{L}}{\partial\mathcal{I}}\boldsymbol{B} \tag{5}$$

where the factor of -4 is inserted for later convenience.

Since $\nabla \times \boldsymbol{H} = 0$, we take $\boldsymbol{H} = -\nabla\phi$, where ϕ is the magnetic scalar potential. Furthermore, the field $\boldsymbol{B}$ is derived from a vector potential (i.e. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$) and so we also have,

$$\nabla \cdot \boldsymbol{B} = 0. \tag{6}$$

If the relationship between $\boldsymbol{H}$ and $\boldsymbol{B}$ were linear, this equation would be satisfied by $\nabla^2\phi = 0$. However, we will assume a small non-linearity between the two fields,

$$-4\frac{\partial\mathcal{L}}{\partial\mathcal{I}} = \mu_0' + \mu_1'(B^2) \tag{7}$$

where μ_0' is a constant and μ_1' a function such that $\mu_1'(B^2) \ll \mu_0'$.

We can invert the relationship between the two fields to first order

$$\boldsymbol{B} = (\mu_0 + \mu_1(H^2))\boldsymbol{H}. \tag{8}$$

Now we recast the field equation with the magnetic potential

$$\nabla \cdot [(\mu_0 + \mu_1(H^2))\nabla\phi] = 0 \tag{9}$$

write $\phi = \phi_0 + \phi_1$, and solve the equation order by order

$$\nabla^2\phi_0 = 0 \tag{10}$$

$$\nabla^2\phi_1 = \rho_{\text{eff}} = -\nabla \cdot \left[\frac{\mu_1(H^2)}{\mu_0}\nabla\phi_0\right] = -2\frac{\mu_1^{(1)}(H^2)}{\mu_0}\nabla\phi_0 \cdot (\nabla\phi_0 \cdot \nabla)\nabla\phi_0 \tag{11}$$

where

$$\mu_1^{(1)}(x) = \frac{\mathrm{d}\mu(x)}{\mathrm{d}x}. \tag{12}$$

For a magnetic dipole,

$$\phi_0(\boldsymbol{r}) = \frac{\boldsymbol{m} \cdot \boldsymbol{r}}{|\boldsymbol{r}|^3} \tag{13}$$

$$\nabla\phi_0 \cdot (\nabla\phi_0 \cdot \nabla)\nabla\phi_0 = 3\frac{[5(\boldsymbol{m} \cdot \boldsymbol{r})^2 + 3|\boldsymbol{m}|^2|\boldsymbol{r}|^2]\boldsymbol{m} \cdot \boldsymbol{r}}{|\boldsymbol{r}|^{13}} \tag{14}$$

or more conveniently in spherical coordinates where we have taken the dipole moment $\boldsymbol{m}$ to be aligned along the z-axis,

$$\phi_0(\boldsymbol{r}) = \sqrt{\frac{4\pi}{3}}\frac{m}{r^2}Y_{10}(\theta, \phi) \tag{15}$$

$$\nabla\phi_0 \cdot (\nabla\phi_0 \cdot \nabla)\nabla\phi_0 = 12\sqrt{\pi}\frac{m^3}{r^{10}}\left[\frac{1}{\sqrt{7}}Y_{30}(\phi, \theta) + \sqrt{3}Y_{10}(\phi, \theta)\right]. \tag{16}$$

3. The Lagrangian to one-loop order

Heisenberg and Euler [5] and Weisskopf [6] independently derived the effective Lagrangian of the electromagnetic field using electron-hole theory. Schwinger [7] later rederived the same result using quantum electrodynamics. In Heaviside–Lorentz units, the Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{4}I \tag{17}$$

$$\mathcal{L}_1 = \frac{e^2}{hc}\int_0^\infty \mathrm{e}^{-\zeta}\frac{\mathrm{d}\zeta}{\zeta^3}\left\{\mathrm{i}\zeta^2\frac{\sqrt{-K}}{4} \times \frac{\cos\left(\frac{\zeta}{B_k}\sqrt{-\frac{I}{2} + \mathrm{i}\frac{\sqrt{-K}}{2}}\right) + \cos\left(\frac{\zeta}{B_k}\sqrt{-\frac{I}{2} - \mathrm{i}\frac{\sqrt{-K}}{2}}\right)}{\cos\left(\frac{\zeta}{B_k}\sqrt{-\frac{I}{2} + \mathrm{i}\frac{\sqrt{-K}}{2}}\right) - \cos\left(\frac{\zeta}{B_k}\sqrt{-\frac{I}{2} - \mathrm{i}\frac{\sqrt{-K}}{2}}\right)}\right.$$

$$\left. + |B_k|^2 + \frac{\zeta^2}{6} I \right\} \tag{18}$$

where $B_k = E_k = \frac{m^2 c^3}{e\hbar} \approx 2.2 \times 10^{15}$ V cm^{-1} $\approx 4.4 \times 10^{13}$ G. Dittrich and Reuter [8] have derived the second-order corrections to the Lagrangian and found them to be in general an order of α smaller than the one-loop corrections regardless of field strength; consequently, the one-loop correction should be adequate for all but the most precise analyses.

In the weak-field limit Heisenberg and Euler [5] give

$$\mathcal{L} \approx -\frac{1}{4} I + E_k^2 \frac{e^2}{hc} \left[\frac{1}{E_k^4} \left(\frac{1}{180} I^2 - \frac{7}{720} K \right) + \frac{1}{E_k^6} \left(\frac{13}{5040} K I - \frac{1}{630} I^3 \right) \cdots \right]. \tag{19}$$

We define a dimensionless parameter ξ to characterize the field strength

$$\xi = \frac{1}{E_k} \sqrt{\frac{I}{2}} \tag{20}$$

and use the analytic expression of this Lagrangian for $K = 0$ derived by Heyl and Hernquist [4]:

$$\mathcal{L}_1(I, 0) = \frac{e^2}{hc} \frac{I}{2} X_0 \left(\frac{1}{\xi} \right) \tag{21}$$

where

$$X_0(x) = 4 \int_0^{x/2-1} \ln(\Gamma(v + 1)) \, dv + \frac{1}{3} \ln \left(\frac{1}{x} \right) + 2 \ln 4\pi - \left(4 \ln A + \frac{5}{3} \ln 2 \right)$$
$$- \left[\ln 4\pi + 1 + \ln \left(\frac{1}{x} \right) \right] x + \left[\frac{3}{4} + \frac{1}{2} \ln \left(\frac{2}{x} \right) \right] x^2 \tag{22}$$

where

$$\ln A = \tfrac{1}{12} - \zeta^{(1)}(-1). \tag{23}$$

Here $\zeta^{(1)}(x)$ denotes the first derivative of the Riemann zeta function.

With the analytic form of the Lagrangian, calculating μ_0' and μ_1' is straightforward and α provides a convenient ordering parameter

$$\mu_0' = -4 \frac{\partial \mathcal{L}_0}{\partial I} = 1 \tag{24}$$

$$\mu_1' = -\frac{\alpha}{2\pi} \left[2X_0 \left(\frac{1}{\xi} \right) - \frac{1}{\xi} X_0^{(1)} \left(\frac{1}{\xi} \right) \right] \tag{25}$$

where $\xi = B/B_k$, and

$$X_0^{(1)}(x) = \frac{dX_0(x)}{dx}. \tag{26}$$

Inverting this relationship to first order yields,

$$\mu_0 = 1 \tag{27}$$

$$\mu_1(H^2) = \frac{\alpha}{2\pi} \left[2X_0 \left(\frac{B_k}{H} \right) - \frac{B_k}{H} X_0^{(1)} \left(\frac{B_k}{H} \right) \right] \tag{28}$$

$$\mu_1^{(1)}(H^2) = \frac{\alpha}{2\pi} \frac{1}{2B_k^2} \left[\frac{B_k^4}{H^4} X_0^{(2)} \left(\frac{B_k}{H} \right) - \frac{B_k^3}{H^3} X_0^{(1)} \left(\frac{B_k}{H} \right) \right] \tag{29}$$

$$= \frac{\alpha}{2\pi} \frac{1}{6B_k^2} \left\{ 2\frac{B_k^2}{H^2} + 3\frac{B_k^3}{H^3} \left[\ln(4\pi) + 1 + \ln\left(\frac{H}{B_k}\right) - 2\ln\Gamma\left(\frac{1}{2}\frac{B_k}{H}\right) \right] \right.$$

$$\left. + 3\frac{B_k^4}{H^4} \left[\psi\left(\frac{1}{2}\frac{B_k}{H}\right) - 1 \right] \right\} \tag{30}$$

where $\psi(x)$ is the digamma function,

$$\psi(x) = \frac{\mathrm{d}\ln\Gamma(x)}{\mathrm{d}x}. \tag{31}$$

The expression for μ_1 agrees numerically with the results of Mielnieczuk *et al* [3]. The function $\mu_1^{(1)}(H^2)$ may conveniently be expanded in the weak-field limit ($H < B_k/2$),

$$\mu_1^{(1)}(H^2) = -\frac{\alpha}{2\pi} \frac{8}{B_k^2} \sum_{j=0}^{\infty} \frac{2^{2j} B_{2(j+2)}}{2j+3} \left(\frac{H}{B_k}\right)^{2j} \tag{32}$$

where B_j denotes the jth Bernoulli number, and in the strong-field limit ($H > B_k/2$)

$$\mu_1^{(1)}(H^2) = \frac{\alpha}{2\pi} \frac{1}{B_k^2} \left\{ \frac{1}{3}\frac{B_k^2}{H^2} - \frac{1}{2}\frac{B_k^3}{H^3} \left[\ln\left(\frac{H}{B_k}\right) + 1 - \ln\pi \right] - \frac{1}{2}\frac{B_k^4}{H^4} \right.$$

$$\left. - \sum_{j=5}^{\infty} \frac{(-1)^j}{2^{j-3}} \frac{j-4}{j-3} \zeta(j-3) \left(\frac{H}{B_k}\right)^{-j} \right\} \tag{33}$$

where we have used the expansions of [4].

As apparent from figure 1, $\mu_1^{(1)}(H^2)$ is constant up to approximately $H = 0.5B_k$ and then begins to decrease quickly as H^{-2}. The existence of these two regimes allows us to find analytic solutions for the correction to the potential (ϕ_1).

4. Solving for the first-order correction

4.1. Weak-field limit

$\mu_1^{(1)}(H^2)$ is constant as long as $H \ll B_k$. In this regime equation (11) may be solved analytically. Since spherical harmonics are eigenfunctions of the angular component of the
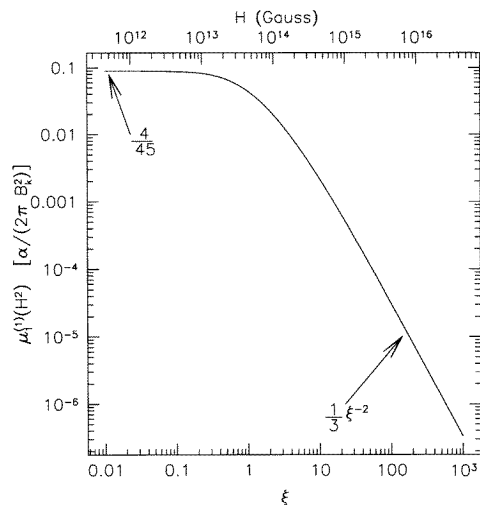
Figure 1. $\mu_1^{(1)}(H^2)$ as a function of ξ, H.

Laplacian operator, it is expedient to expand the right-hand side of equation (11) in terms of spherical harmonics [9].

$$\rho_{lm}(r) = \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi \, Y_{lm}^*(\theta, \phi) \rho_{\text{eff}}(r, \theta, \phi). \tag{34}$$

With this definition the correction to the potential is given by [9]

$$\phi_1(r, \theta, \phi) = -\sum_{l,m} \frac{Y_{lm}(\theta, \phi)}{2l+1} \left[\frac{1}{r^{l+1}} \int_0^r \rho_{lm}(a) a^{l+2} \, da + r^l \int_r^\infty \rho_{lm}(a) \frac{da}{a^{l-1}} \right]. \tag{35}$$

If $\mu_1^{(1)}(H^2)$ is a constant, we see from equations (11) and (16) that the expansion given by equation (34) is straightforward

$$\rho_{10}(r) = -24\sqrt{3\pi} \frac{\mu_1^{(1)}}{\mu_0} \frac{m^3}{r^{10}} \tag{36}$$

$$\rho_{30}(r) = -24\sqrt{\frac{\pi}{7}} \frac{\mu_1^{(1)}}{\mu_0} \frac{m^3}{r^{10}}. \tag{37}$$

If we combine these results with equation (35) we see that the interior integral diverges if $\mu_1^{(1)}$ is a constant. This does not present a problem if we insert an inner bound (r_0) to the interior integral. This inner bound is defined as the radius at which either $\mu_1^{(1)}(H^2)$ begins to change (i.e. when $H \gtrsim 0.5B_k$) or when the zeroth order potential is no longer given by the dipole formula (i.e. at the surface of the object). We obtain

$$\phi_{1,10}(r, \theta, \phi) = \frac{4}{9}\sqrt{3\pi} \frac{\mu_1^{(1)}}{\mu_0} \frac{m^3}{r^2} \left(\frac{3}{r_0^6} - \frac{1}{r^6} \right) Y_{10}(\theta, \phi) \tag{38}$$

$$\phi_{1,30}(r, \theta, \phi) = 6\sqrt{\frac{\pi}{7}} \frac{\mu_1^{(1)}}{\mu_0} \frac{m^3}{r^4} \left(\frac{1}{7} \frac{1}{r_0^4} - \frac{1}{11} \frac{1}{r^4} \right) Y_{30}(\theta, \phi). \tag{39}$$

These functions describe radially dependent corrections to the dipole and hexapole moments of the object under consideration. If we define the higher moments in analogy to equation (15),

$$\phi_{l0}(r, \theta, \phi) = \sqrt{\frac{4\pi}{2l+1}} \frac{M_{l0}}{r^{(l+1)}} Y_{l0}(\theta, \phi) \tag{40}$$

we obtain the following corrections

$$m_1(r) = \frac{2}{3} \frac{\mu_1^{(1)}}{\mu_0} m^3 \left(\frac{3}{r_0^6} - \frac{1}{r^6} \right) \tag{41}$$

$$M_{30,1}(r) = 3 \frac{\mu_1^{(1)}}{\mu_0} m^3 \left(\frac{1}{7} \frac{1}{r_0^4} - \frac{1}{11} \frac{1}{r^4} \right). \tag{42}$$

To eliminate the dependence on the inner bound, r_0, we assume that the dipole and hexapole moments are known at a radius $r_s > r_0$ and calculate the difference between the known moments at r_s and those measured at infinity.

Substituting the values of μ_0 and $\mu_1^{(1)}$, we obtain,

$$m(r = \infty) - m(r = r_s) = \Delta m = \frac{8}{135} \frac{\alpha}{2\pi} m \left(\frac{m}{r_s^3} \frac{1}{B_k} \right)^2 \tag{43}$$

$$M_{30}(r = \infty) - M_{30}(r = r_s) = \Delta M_{30} = \frac{4}{165} \frac{\alpha}{2\pi} m r_s^2 \left(\frac{m}{r_s^3} \frac{1}{B_k} \right)^2. \tag{44}$$

Both the corrections are given in terms of the magnetic field strength at r_s. Since we have assumed that $\mu_1^{(1)}(H^2)$ is constant near r_s, the field strength at r_s must appreciably be less than B_k for this set of approximations to be valid. Consequently, the correction to the dipole moment is indeed quite small, less than one-thousandth of the 'bare' dipole moment. However, there is a correction to the hexapole moment even without a 'bare' hexapole moment. Thus radiative corrections generate a hexapole field which is in principle measurable at infinity.

4.2. The general case

In the strong-field limit, we must use the general equation (30) for $\mu_1^{(1)}(H^2)$ because the magnetic field strength varies as a function of θ around the dipole. Before tackling this problem numerically, we can glean several characteristics of the solutions from equations (11) and (34). Since for a dipole ρ_{eff} is an odd function of θ and constant with respect to ϕ, the contributions,

$$\rho_{lm}(r) = 0 \qquad \text{if } l \text{ is even or } m \neq 0. \tag{45}$$

Furthermore, we can more conveniently write

$$\rho_{lm}(r) = \frac{1}{\mu_0} \frac{\alpha}{2\pi} \frac{1}{B_k^2} \frac{m^3}{r^{10}} \chi_{lm}(\beta) \tag{46}$$

where $\chi_{lm}(\beta)$ is a dimensionless function of a dimensionless argument,

$$\beta = \frac{m}{r^3} \frac{1}{B_k}. \tag{47}$$

For a dipole,

$$\chi_{lm}(\beta) = -2 \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\phi Y_{lm}^*(\theta, \phi) \left(\frac{\alpha}{2\pi} \frac{1}{B_k^2} \right)^{-1}$$
$$\times (3\cos\theta(5\cos^2\theta + 3)) \mu_1^{(1)}[\beta^2 B_k^2(3\cos^2\theta + 1)]. \tag{48}$$

We calculate numerically the functions $\chi_{lm}(\beta)$ for the first three odd harmonics and depict the results in figure 2.

The limiting expressions for the weak-field limit are easily calculated,

$$\chi_{10}(\beta) = \sqrt{3\pi}[-\tfrac{32}{15} + \tfrac{13312}{1225}\beta^2 - \tfrac{167936}{1225}\beta^4 + \mathcal{O}(\beta^6)] \tag{49}$$

$$\chi_{30}(\beta) = \sqrt{7\pi}[-\tfrac{32}{105} + \tfrac{11776}{3675}\beta^2 - \tfrac{323584}{5775}\beta^4 + \mathcal{O}(\beta^6)] \tag{50}$$

$$\chi_{50}(\beta) = \sqrt{11\pi}[+\tfrac{512}{1617}\beta^2 - \tfrac{1150976}{105105}\beta^4 + \mathcal{O}(\beta^6)]. \tag{51}$$

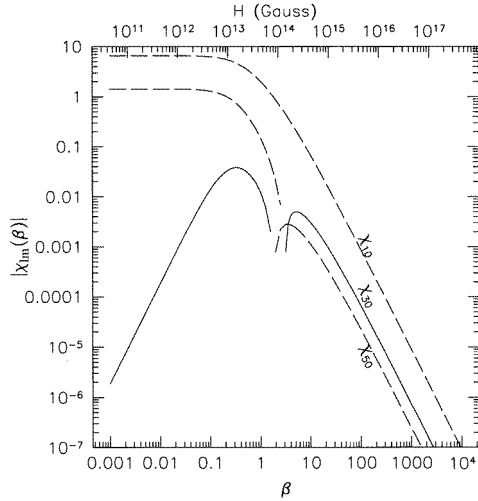In the strong-field limit, we have the following approximations

$$\chi_{10}(\beta) = -\tfrac{4}{27}\sqrt{\pi}(27\sqrt{3} - 4\pi)\beta^{-2} + \mathcal{O}(\beta^{-3}) \tag{52}$$

$$\chi_{30}(\beta) = \tfrac{8}{243}\sqrt{7\pi}(81 - 14\sqrt{3}\pi)\beta^{-2} + \mathcal{O}(\beta^{-3}) \tag{53}$$

$$\chi_{50}(\beta) = -\tfrac{4}{1215}\sqrt{11\pi}(1863 - 340\sqrt{3}\pi)\beta^{-2} + \mathcal{O}(\beta^{-3}). \tag{54}$$

The integrals for each spherical-harmonic component of the first-order correction (equation (35)) may be recast in terms of integrals over β

$$\phi_1(r, \theta, \phi) = -\frac{1}{3} \frac{m}{r^2} \frac{1}{\mu_0} \frac{\alpha}{2\pi} \beta^2 \sum_{l,m} \frac{Y_{lm}(\theta, \phi)}{2l + 1}$$
$$\times \left[\beta^{(l-7)/3} \int_\beta^{\beta_0} \chi_{lm}(v) v^{-(l-4)/3} \, dv + \beta^{-(l+8)/3} \int_0^\beta \chi_{lm}(v) v^{(l+5)/3} \, dv \right] \tag{55}$$

Figure 2. $\chi_{lm}(\beta)$ as a function of β, H. The full curves indicate where $\chi_{lm}(\beta)$ is positive. The broken curves indicate negative values of the ordinate.

where the first integral is to be evaluated in the limit as $\beta_0 \to \infty$. Although equation (55) appears to be scale free, the cut-off β_0 has a physical interpretation. First, it can be taken to be the magnetic field strength at the surface of the object. For a point magnetic dipole (e.g. an electron), the interpretation is more subtle. As one approaches a point dipole, not only does the field become arbitrarily strong, so do the field gradients. When these gradients become larger in magnitude than B_k/λ_e ($\lambda_e = \hbar/m_e c$, the electron Compton wavelength), the Heisenberg–Euler Lagrangian is no longer applicable; therefore, we do not expect our expressions for ρ_{eff} to be valid arbitrarily close to a point dipole. The radius or field strength at which our expression for ρ_{eff} fails depends on the intrinsic dipole moment of the object (m),

$$r_0 \sim \left(\frac{m}{B_k}\lambda_e\right)^{1/4} \qquad \text{or} \qquad \beta_0 \sim \left(\frac{m}{B_k}\right)^{1/4}\lambda_e^{-3/4}. \tag{56}$$

As in the weak-field case, we calculate the shift in the observed multipole moments at infinity relative to the known moments at some inner radius $r_s > r_0$ or equivalently at some field strength $\beta_s < \beta_0$.

$$m_1(0) - m_1(\beta_s) = -\frac{1}{9}m\frac{1}{\mu_0}\frac{\alpha}{2\pi}\sqrt{\frac{3}{4\pi}}\left[\int_0^{\beta_s}\chi_{10}(v)v\,\mathrm{d}v - \frac{1}{\beta_s}\int_0^{\beta_s}\chi_{10}(v)v^2\,\mathrm{d}v\right] \tag{57}$$

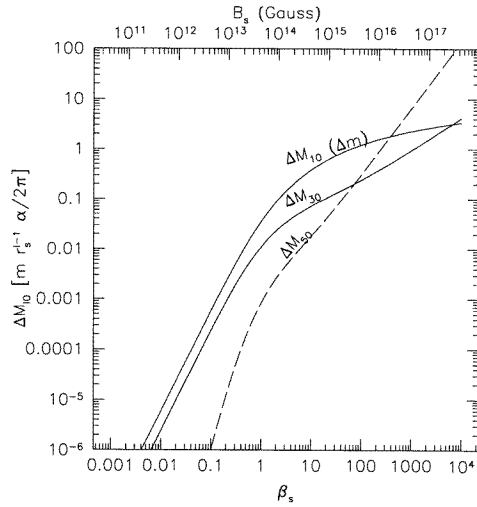$$M_{30,1}(0) - M_{30,1}(\beta_s) = -\frac{1}{21}mr_s^2\frac{1}{\mu_0}\frac{\alpha}{2\pi}\sqrt{\frac{7}{4\pi}}$$

$$\times\left[\beta_s^{2/3}\int_0^{\beta_s}\chi_{30}(v)v^{1/3}\,\mathrm{d}v - \beta_s^{-5/3}\int_0^{\beta_s}\chi_{30}(v)v^{8/3}\,\mathrm{d}v\right] \tag{58}$$

$$M_{50,1}(0) - M_{50,1}(\beta_s) = -\frac{1}{33}mr_s^4\frac{1}{\mu_0}\frac{\alpha}{2\pi}\sqrt{\frac{11}{4\pi}}$$

$$\times\left[\beta_s^{4/3}\int_0^{\beta_s}\chi_{50}(v)v^{-1/3}\,\mathrm{d}v - \beta_s^{-7/3}\int_0^{\beta_s}\chi_{50}(v)v^{10/3}\,\mathrm{d}v\right] \tag{59}$$

and in general

$$M_{l0,1}(0) - M_{l0,1}(\beta_s) = -\frac{1}{3(2l+1)}mr_s^{l-1}\frac{1}{\mu_0}\frac{\alpha}{2\pi}\sqrt{\frac{2l+1}{4\pi}}$$

Figure 3. The difference in the strength of the multipole moments measured at infinity ($\beta \to 0$) and at the surface, $M_{l0}(0) - M_{l0}(\beta_s)(\Delta M_{l0})$, as a function of the strength of the dipole field at the surface, β_s, H_s. The full curves indicate the moments for which the vacuum acts paramagnetically, i.e. $\Delta M_{l0}(\beta_s) > 0$. The broken curve indicates negative values of the ordinate.

$$\times \left[\beta_s^{(l-1)/3} \int_0^{\beta_s} \chi_{l0}(v) v^{-(l-4)/3} \, \mathrm{d}v - \beta_s^{-(l+2)/3} \int_0^{\beta_s} \chi_{l0}(v) v^{(l+5)/3} \, \mathrm{d}v \right]. \qquad (60)$$

Figure 3 depicts the shifts in the moments between the surface of the dipole and infinity.

For the higher moments ($l > 1$), the integrals in equation (55) are well behaved so we need not set an inner bound (β_0) and we can define in general

$$M_{l0,1}(0) = -\frac{1}{3(2l+1)} m \left(\frac{m}{B_k} \right)^{(l-1)/3} \frac{1}{\mu_0} \frac{\alpha}{2\pi} \sqrt{\frac{2l+1}{4\pi}} \int_0^\infty \chi_{l0}(v) v^{-(l-4)/3} \, \mathrm{d}v. \qquad (61)$$

This integral converges for all $l > 1$. For $l = 1$, it diverges logarithmically as $v \to \infty$.

5. Conclusions: Application to neutron stars

The environment of a neutron star is strongly magnetized; therefore, the one-loop QED corrections may be significant, especially for those neutron stars with ultrastrong surface fields (magnetars) [10]. Although plasma probably fills this region, QED vacuum corrections dominate the contribution of the plasma to the magnetic permeability [11].

The structure of the magnetic field at the surface of a neutron star is an important clue to the origin of neutron-star magnetic fields. Several authors have proposed [12] that currents in the thin crust generate the observed magnetic fields. In this case, the field structure will be dominated by high-order multipoles with $l \sim \delta r_c / r_* \gg 1$ where δr_c is the thickness of the crust and r_* is the radius of the star [13]. Arons [13] argues further that the observed spin-up line for millisecond pulsars constrains the strength of higher-order multipoles at the surface to be no more than 40% of surface strength of the dipole.

The current results complicate this argument. The location of the observed spin-up line and the value of the spin-down index of a pulsar depend on the strength of the various moments of the magnetic field at the light cylinder. Our results show that the vacuum itself may generate higher magnetic moments between the neutron star surface and the light cylinder. Figure 4 depicts the fractional contribution of higher magnetic moments to the field strength at the light cylinder as a function of the surface dipole field strength and the pulsar period.
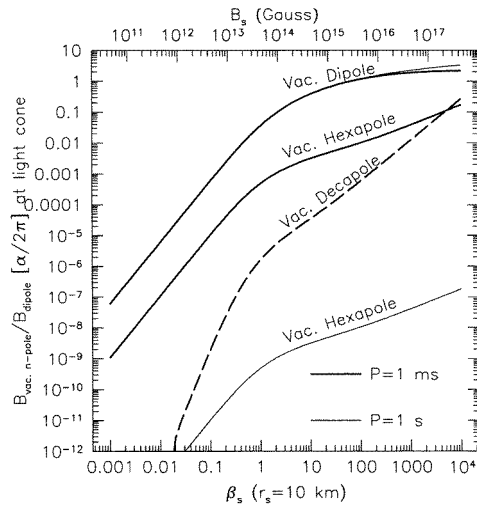
Figure 4. The fractional contribution of higher magnetic moments to the field strength at the light cylinder as a function of the surface dipole field strength for periods of 1 ms and 1 s. The contribution of the vacuum decapole at the light cone for $P = 1$ s is too small to be depicted on this graph.

Even for magnetars near their birth, the vacuum adds only a small correction to the field strength at the light cone. Because of the weakness of the QED coupling, the one-loop corrections to otherwise classical descriptions of a magnetic dipole tend to be small for all but the most extreme field strengths.

Observing this effect would be difficult. For a magnetar, measurements of the field strength at two different radii and an estimate of the strength of higher-order moments at the surface each to a precision of one part in one-thousand would be required. However, if one could argue that an object had no hexapole or higher-order multipole intrinsically, a measurement of a higher-order multipole far from the object would uncover the effects of one-loop corrections. For example, an electron is intrinsically a magnetic dipole. If one ignores the contribution of terms in the Lagrangian depending on field gradients, one would expect the vacuum surrounding an electron to generate higher-order multipole fields. Determining whether this occurs is beyond the scope of this work.

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